MATH 2050C Lecture 11 (Feb 22)

Announcement about Midterm

- Format: Take-home, open book / notes
- Time: Mar 3, 2022 8:30 AM - 10:00 AM
- Coverage: Lecture 1-11, up to § 3.3 inclusively
  (i.e. Problem Set 1-6)

Last week: limit theorems, squeeze thm, ratio test

Q: Can we decide the limit of (x_n) exists or not WITHOUT "knowing" the value of the limit?

Recall: (ε-K def$^{\text{n}}$) $\lim(x_n) = x$

iff $\forall \varepsilon > 0, \exists K \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon \quad \forall n \geq K$

Recall: (x_n) convergent $\overset{\not\Leftarrow}{\Rightarrow}$ (x_n) bdd

False $\because$ $((-1)^n)$ is a divergent but bdd sequence

Question: IF we are willing to assume more about the seq. (x_n), does

(x_n) bdd $\Rightarrow$ (x_n) convergent?

Answer: YES! provided the seq is "monotone"

Monotone Convergence Theorem (MCT)

(x_n) bdd + monotone $\Rightarrow$ (x_n) convergent

Def$^{\underline{n}}$: (x_n) is monotone if it is

either (i) increasing, i.e. $x_1 \le x_2 \le x_3 \le \cdots$, $(x_n \le x_{n+1} \;\forall n \in \mathbb{N})$

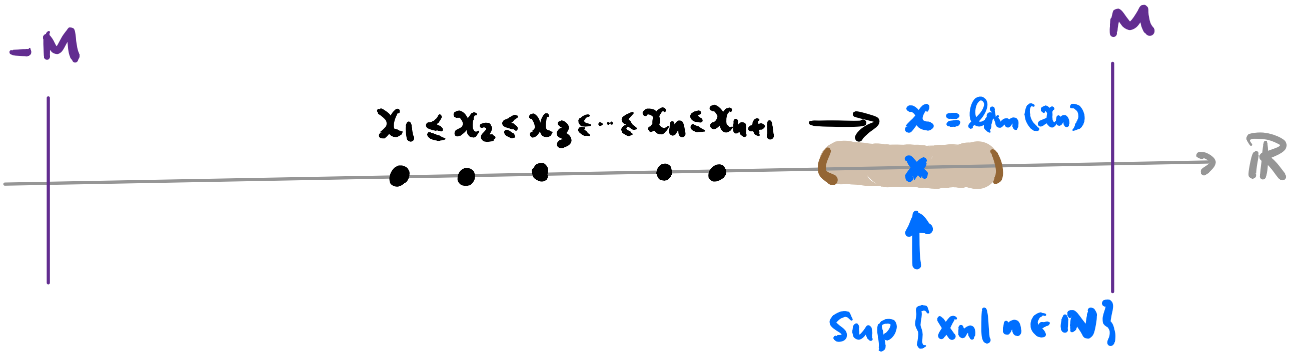
or (ii) decreasing, i.e. $x_1 \ge x_2 \ge x_3 \ge \cdots$, $(x_n \ge x_{n+1} \;\forall n \in \mathbb{N})$

Remark: Whenever the inequalities above are all strict inequalities, we say that the seq. is strictly monotone/ increasing / decreasing respectively.

Remark: A convergent seq. may NOT be monotone!

E.g. $\left((-1)^n \cdot \frac{1}{n} \right) \to 0$ but not monotone

Picture: (x_n) increasing & bdd (by $M > 0$)



Proof of Theorem:

IDEA: Expect $\lim(x_n) = \sup\{x_n \mid n \in \mathbb{N}\}$!

Suppose (x_n) is bdd & increasing. Consider

$$\phi \neq S := \{x_n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$$

- (x_n) bdd $\underset{\text{def}^{\text{n}}}{\Longleftrightarrow}$ S is bdd as a subset of $\mathbb{R}$

$\Rightarrow$ S is bdd from above

[Note: An increasing seq is always bdd from below.]

By Completeness Property of $\mathbb{R}$,

$\boxed{x := \sup S} \in \mathbb{R}$ exists !

Claim: $\lim(x_n) = x$

Pf of Claim: We verify the ε-K definition.

Let $\varepsilon > 0$ be fixed but arbitrary.

Since x is the supremum of S, the number

$x - \varepsilon$ CANNOT be an upper bound of S

i.e. $\exists\, K \in \mathbb{N}$ s.t. $\quad x - \varepsilon < x_K$

Observe that (x_n) is increasing by assumption

$$\Rightarrow \quad x-\varepsilon < x_K \le x_{K+1} \le x_{K+2} \le \cdots \le x_n, \quad \forall n \ge K$$

On the other hand, since $x = \sup S$ is an upper bound of S, we have

$$\Rightarrow \quad x_n \le x < x+\varepsilon \quad \forall n \in \mathbb{N}$$

Combining the two inequalities above, we obtain

$$\forall n \ge K, \quad x-\varepsilon < x_n < x+\varepsilon$$

$\blacksquare$

Remark: Actually, we have proved the following:

(x_n) increasing & bdd above

$$\Rightarrow \quad \lim(x_n) = \underline{\sup\{x_n \mid n \in \mathbb{N}\}}$$

(x_n) decreasing & bdd below

$$\Rightarrow \quad \lim(x_n) = \underline{\inf\{x_n \mid n \in \mathbb{N}\}}$$

We also "know" the the limit as well.

In summary, MCT says that

$$(x_n) \text{ bdd} \iff (x_n) \text{ convergent}$$

provided that (x_n) is monotone (Caution: $((-1)^n)$)

<u>Example 1</u> : "Harmonic series"

Let $h_n := 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$, $n \in \mathbb{N}$.

i.e. $h_1 = 1$, $h_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $h_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$ ....

Show that (h_n) is divergent.

<u>Proof</u>: Observe that (h_n) is (strictly) increasing

since $h_{n+1} = h_n + \frac{1}{n+1} > h_n \quad \forall n \in \mathbb{N}$

By MCT, (h_n) divergent $\iff$ (h_n) unbdd.

<u>Claim</u>: (h_n) is <u>unbdd</u>.!

<u>Pf</u>: 

$h_1 = 1$, $h_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $h_3 = \frac{11}{6}$

$$h_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$$> \underbrace{1}_{1} + \underbrace{\frac{1}{2}}_{\frac{1}{2}} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}}$$

Consider $n = 2^m$ for $m \in \mathbb{N}$.

$$h_{2^m} = \underbrace{1}_{} + \underbrace{\tfrac{1}{2}}_{1 \text{ term}} + \underbrace{\left(\tfrac{1}{3} + \tfrac{1}{4}\right)}_{2 \text{ terms}} + \underbrace{\cdots}_{4 \text{ terms}} + \underbrace{\left(\tfrac{1}{2^{m-1}+1} + \cdots + \tfrac{1}{2^m}\right)}_{2^{m-1} \text{ terms}}$$

$$> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \cdots + \left(\tfrac{1}{2^m} + \cdots + \tfrac{1}{2^m}\right)$$

$$= 1 + \underbrace{\tfrac{1}{2} + \tfrac{1}{2} + \cdots + \tfrac{1}{2}}_{m \text{ terms}} = 1 + \frac{m}{2}$$

becomes unbdd as $m \to \infty$

So (h_n) is unbdd, hence divergent. ▫

Remark: MCT works well particularly for "recursively defined" sequences

General Strategy to do this:

Step 1: Apply MCT to show the limit exists

Step 2: Take limit in the recursive relation (*) to compute the desired limit

Example 2: Define the seq. (y_n) recursively by

$$y_1 = 1, \quad \boxed{y_{n+1} := \frac{1}{4}(2y_n + 3)} \ (*) \quad \forall n \in \mathbb{N}$$

Show that $\lim(y_n) = 3/2$.

Note: $y_1 = 1$, $y_2 = \frac{1}{4}(2\cdot 1 + 3) = \frac{5}{4}$

$y_3 = \frac{1}{4}(2\cdot\frac{5}{4} + 3) = \frac{11}{8}$

<u>Proof:</u> We first show that (y_n) is bdd & monotone.

<u>Claim</u>: (y_n) is bdd above by 2

<u>Pf of Claim</u>: Use M.I. to show $y_n \leq 2 \quad \forall n \in \mathbb{N}$.

$n=1$. $\quad y_1 = 1 < 2$ $\quad$ Done.

Assume $y_k \leq 2$, then $\quad y_{k+1} = \frac{1}{4}(2y_k+3) \leq \frac{7}{4} < 2$

<u>Claim</u>: (y_n) is increasing, i.e. $y_n \leq y_{n+1} \quad \forall n \in \mathbb{N}$.

<u>Pf of Claim</u>: Use M.I. again.

$n=1$: $\quad y_1 := 1 < \frac{5}{4} = y_2$ $\quad$ Done!

Assume $y_k \leq y_{k+1}$. Then

$$y_{k+1} = \frac{1}{4}(2y_k+2) \leq \frac{1}{4}(2y_{k+1}+2) = y_{k+2}$$

Combining these two claims, MCT $\Rightarrow \lim(y_n) =: y$ exists.

Since (y_n) is convergent, we have

$$\lim_{n\to\infty}(y_{n+1}) = \lim_{n\to\infty}(y_n) = y$$

Taking limit on both sides of (*).

$$\lim(y_{n+1}) = \lim \frac{1}{4}(2y_n+3) = \frac{1}{4}(2\lim(y_n)+3)$$

Limit Thm $\because \lim(y_n)$ exists

we obtain an equation:

$$y = \frac{1}{4}(2y+3) \xrightarrow[\text{for } y]{\text{solve}} y = 3/2$$

$\square$

Example 3: Fix $a > 0$. Define inductively

$$S_1 := 1 \quad ; \quad \boxed{S_{n+1} := \frac{1}{2}\left(S_n + \frac{a}{S_n}\right)} \quad (**) \quad \forall n \in \mathbb{N}$$

Ex: Try to generate the first few terms for $a = 2$.

Show that $\lim(S_n) = \sqrt{a} > 0$.

Proof: We follow the same general strategy.

Claim 1: (S_n) bdd below by $\sqrt{a}$ (for $n \geq 2$)

Pf of Claim: Note $S_n > 0 \ \forall n \in \mathbb{N}$. Rewrite $(**)$ as

$$S_n^2 - 2S_{n+1}S_n + a = 0$$

So, the quadratic eq$^{\underline{n}}$ $x^2 - 2S_{n+1}x + a = 0$ has at least one real root, namely $x = S_n$,

$$\Rightarrow \quad 4S_{n+1}^2 - 4a \geq 0$$

$$\Rightarrow \quad S_{n+1} \geq \sqrt{a} \qquad \forall n \in \mathbb{N}$$

Claim 2: (S_n) is eventually decreasing

ie $S_n \geq S_{n+1} \quad \forall n \geq 2$

Pf of Claim: $\forall n \geq 2$,

Claim 1

$$S_n - S_{n+1} = S_n - \frac{1}{2}\left(S_n + \frac{a}{S_n}\right) = \frac{1}{2}\left(\frac{S_n^2 - a}{S_n}\right) \geq 0$$

By MCT, $\lim(S_n) =: S$ exists.

Take $n \to \infty$ on both sides of (**), we obtain an equation:

$$S = \frac{1}{2}\left(S + \frac{a}{S}\right) \xrightarrow[\text{for } S]{\text{solve}} S = \sqrt{a} \text{ or } -\sqrt{a}$$

rejected

$\because S_n \geq \sqrt{a} > 0$

$\Rightarrow S \geq \sqrt{a} > 0$

$\square$